

Dynamical Instability of Cylindrical Symmetric Collapsing Star in Generalized Teleparallel Gravity

Abdul Jawad¹*, Davood Momeni^{2†}, Shamaila Rani^{1‡}
and Ratbay Myrzakulov^{2§}

¹Department of Mathematics, COMSATS Institute of Information Technology, Lahore-54000, Pakistan.

²Eurasian International Center for Theoretical Physics and Department of General & Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan

Abstract

This paper is devoted to analyze the dynamical instability of a self-gravitating object undergoes to collapse process. We take the framework of generalized teleparallel gravity with cylindrical symmetric gravitating object. The matter distribution is represented by locally anisotropic energy-momentum tensor. We develop basic equations such as dynamical equations along with matching conditions and Harrison-Wheeler equation of state. By applying linear perturbation strategy, we construct collapse equation which is used to accomplish the instability ranges in Newtonian and post-Newtonian regimes. We find these ranges for isotropic pressure as well as reduce the results in general relativity. The unstable behavior depends on matter, metric, mass and torsion based terms.

*jawadab181@yahoo.com; abduljawad@ciitlahore.edu.pk

†d.momeni@yahoo.com

‡drshamailarani@ciitlahore.edu.pk

§rmyrzakulov@gmail.com

Keywords: $f(T)$ gravity; Instability; Cylindrical symmetry; Newtonian and post-Newtonian regimes.

PACS: 04.50.Kd; 04.25.Nx; 04.40.Dg.

1 Introduction

The generalized teleparallel theory of gravity ($f(T)$ where T is torsion scalar) is one of the modified theories which occupied a vast area of research in modern cosmology and Astrophysics. General relativity is identified as the cornerstone of modern cosmology. Einstein introduced cosmological constant, a simplest model which is marked by cosmological observations on the basis of general relativity (GR). This theory based on curvature via Levi-Civita connection. This constant happened to inherit some flaws later like fine-tuning and cosmic coincidence problems. Later on, Einstein proposed a theory which is equivalent to GR established through parallel transported of vierbein field, named as teleparallel gravity. In this theory, the torsion scalar laid down the gravitational field taking into account Weitzenböck connection. The modified form of GR shapes $f(R)$ gravity while $f(T)$ is the modification of teleparallel gravity by modifying curvature and torsion scalars upto higher order terms.

Firstly, the $f(T)$ theory of gravity is proposed under Born-Infeld strategy which helped to solve particle horizon problem and found singularity-free solution (Ferraro and Fiorini 2008). Later, many phenomena which includes expanding universe with acceleration through different cosmological parameters, planes, perturbations, cosmographic techniques, energy conditions, spherical symmetric solutions via solar system constraints, Birkhoff's theorem, connection between different regions of the universe through static and wormhole solutions, viability of thermodynamical laws, unstable behavior of spherical symmetric collapsing stars applying different dynamical conditions, reconstruction scenario via dynamical dark energy models, (Jamil et al. 2012; Jamil, Momeni & Myrzakulov 2012, 2013, Houndjo, Momeni & Myrzakulov 2012; Wang 2011a,b; Daouda, Rodrigues & Houndjo 2011, 2012; Gonzalez, Saridakis and Vsquez 2012; Bamba, Nojiri & Odintsov 2014; Sharif and Rani 2013, 2014a,b; Jawad and Rani 2015) etc. The $f(T)$ theory of gravity is non local Lorentz invariant theory. To resolve this problem, a lot of work has been done in this direction (Li, Sotiriou & Barrow 2011; Li, Miao & Miao 2011). Nashed (2015) proposed general tetrad field by regularization of

$f(T)$ field equations which has two tetrad matrices. This regularized process with general tetrad field removed the effect of local Lorentz invariance.

The dynamics of self-gravitating object during collapse process has been analyzed in GR (Skripkin 1960; Chandrasekhar 1964; Herrera, Santos & Le Denmat 1989; Chan, Herrera & Santos 1993; Herrera & Santos 1995; Herrera, Le Denmat and Santos & 2012) and modified theories of gravity such as $f(R)$, $f(R, T)$, Brans-Dicke, Einstein Gauss-Bonnet gravity in spherical as well as cylindrical symmetry (Sharif & Manzoor 2014, 2015; Kausar, Noureen & Shahzad 2015; Kausar 2013) while in $f(T)$ gravity for spherical symmetry (Sharif & Rani 2014, 2015; Jawad & Rani 2015). The disturbance in hydrostatic equilibrium of a stellar object leads to the collapse process. The instability occurs when the weight of the outer region surrounded the object overcome very quickly the pressure inside the object and gravitational force consequently pushes the matter towards center of the object initiating collapse. In order to study the unstable behavior of self-gravitating collapsing objects, the adiabatic index which is a stiffness parameter is used. For spherically symmetric matter configuration of collapsing star, this index gives numerical range of the instability as less than $4/3$ in GR. However, in modified theories of gravity, there appear effective terms results from modification of theory. Now-a-days, a lot of work is done on jeans instability too (Roshan & Abbass 2014, 2015).

In $f(R)$ gravity, Sharif & Kausar (2011) and Kausar & Noureen (2014) analyzed the dynamical instability ranges for Newtonian and post-Newtonian regimes of a spherical symmetric collapsing star with and without charge. These ranges depend on matter, metric, mass and curvature based terms. For cylindrical symmetry, Kausar (2013) studied the effects of CDTT model which has inverse curvature term on the unstable behavior. The asymptotic behavior is also obtained for both regimes. Sharif & Manzoor (2014) examined the instability of cylindrically symmetric collapsing object in the frame work of Brans-Dicke gravity. They concluded that adiabatic index remains less than one for the unstable behavior while greater than one in a special case. In $f(T)$ gravity, Sharif & Rani (2014, 2015) analyzed the dynamics of self-gravitating object with spherical symmetry via expansion and expansion free cases. Recently, Jawad & Rani (2015) examined the instability ranges taking into account shear-free condition for Newtonian and post-Newtonian regimes.

The collapse process happens when stability of matter disturbed and at long last experiences collapse which leads to different structures. We have

taken the self-gravitating object as cylindrically symmetric collapsing star. The dynamical instability analysis is mostly done for spherically symmetric object which includes galactic halos, globular clusters, etc. However, the non-spherical objects such that cylindrical symmetry and plates came into being by post-shocked clouds on the verge of gravitational collapse at stellar scales as well as galaxy formations. The cylindrical symmetry is associated with the problem of fragmentation of pre-stellar clouds. Specifically, final fate of collapse of a non-spherical cloud coming from numerical relativity and certain analytical solutions in cylindrical symmetry provide some new examples about gravitational collapse.

In this connection, we extend our work on dynamical instability to the cylindrically symmetric collapsing stars in $f(T)$ gravity. We analyze the dynamical instability ranges in Newtonian and post-Newtonian regimes. The paper is organized as follows: in next section, we provide the basics of generalized teleparallel gravity. Section 3 is devoted to the construction of some basic equations such as field, dynamical and matching equations. In section 4, we develop collapse equation for cylindrically symmetric collapsing star. The instability ranges for anisotropic as well as isotropic fluid for Newtonian and post-Newtonian regimes are examined in section 5. Last section provides results of the paper.

2 Generalized Teleparallel Gravity

In this section, we provide the basics and formulation of generalized teleparallel gravity. This gravity is defined through the action (Ferraro & Fiorini 2008; Linder 2010a,b; Bamba, Geng & Lee 2010; Wu & Yu 2010a,b, 2011; Bamba et al. 2011; de la Cruz-Dombriz, Dunsby & Saez-Gomez 2014)

$$\mathcal{I} = \frac{1}{\kappa^2} \int d^4x (\mathcal{L}_m + f(T))h, \quad (1)$$

where $f(T)$ is an arbitrary differentiable function, T denotes torsion scalar, \mathcal{L}_m represents the matter density and κ^2 denotes the coupling constant. The term $h = \det(h^a_\beta)$ is the determinant of vierbein (or tetrad) field, h^a_β . This field has a basic and central part in construction of this torsion based gravity. This is an orthonormal set of vector fields related with metric tensor by the relation $g_{\beta\alpha} = \eta_{ab} h^a_\beta h^b_\alpha$, $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is the Minkowski space. It is noted here that the indices (a, b, \dots) are the coordinates of tangent

space while (α, β, \dots) expresses the coordinate indices on manifold and all these indices run from 0, 1, 2, 3. The parallel transported of vierbein field by the significant component Weitzenböck connection in Weitzenböck spacetime leads the basic construction of teleparallel as well as generalized teleparallel gravity where $\tilde{\Gamma}^\beta_{\alpha\gamma} = h_a^\beta \partial_\gamma h^a_\alpha$ is the Weitzenböck connection. We obtain torsion tensor by the antisymmetric part of this connection as follows

$$T^\beta_{\alpha\gamma} = \tilde{\Gamma}^\beta_{\gamma\alpha} - \tilde{\Gamma}^\beta_{\alpha\gamma} = h_a^\beta (\partial_\gamma h^a_\alpha - \partial_\alpha h^a_\gamma), \quad (2)$$

which is antisymmetric in its lower indices, i.e., $T^\beta_{\alpha\gamma} = -T^\beta_{\gamma\alpha}$. It is noted that the parallel transported of tetrad field vanishes rapidly the curvature of the Weitzenböck connection.

The torsion scalar takes the form

$$T = S_\beta^{\alpha\gamma} T^\beta_{\alpha\gamma}. \quad (3)$$

where

$$S_\beta^{\alpha\gamma} = \frac{1}{2} [\delta_\beta^\alpha T^{\mu\gamma}_\mu - \delta_\beta^\gamma T^{\mu\alpha}_\mu - \frac{1}{2} (T^{\alpha\gamma}_\beta - T^{\gamma\alpha}_\beta - T^\beta_{\alpha\gamma})]. \quad (4)$$

Varying of action of $f(T)$ gravity w.r.t vierbein field, we obtain the field equations as

$$h_a^\beta S_\beta^{\alpha\gamma} \partial_\alpha T f_{TT} + \left[\frac{1}{h} \partial_\alpha (h h_a^\beta S_\beta^{\alpha\gamma}) + h_a^\beta T^\lambda_{\alpha\beta} S_\lambda^{\gamma\alpha} \right] f_T + \frac{1}{4} h_a^\gamma f = \frac{1}{2} \kappa^2 h_a^\beta \Theta_\beta^\gamma, \quad (5)$$

where subscript T and TT represent first and second order derivatives of f with respect to T . In terms of covariant derivative instead of partial derivatives, the $f(T)$ field equations can be reconstructed. In covariant formalism, we obtain an important condition, $R = -T - 2\nabla^\beta T^\gamma_{\beta\gamma}$. This implies that the covariant derivative of torsion tensor depicts the only difference between Ricci and torsion scalars.

By implying the strategy of covariant derivative (widely discussed in (Sharif & Rani 2013, 2014a,b; Jawad & Rani 2015)), we get the field equations in $f(T)$ gravity as

$$f_T G_{\alpha\gamma} + \frac{1}{2} g_{\alpha\gamma} (f - T f_T) + \mathcal{D}_{\alpha\gamma} f_{TT} = \kappa^2 \Theta_{\alpha\gamma}, \quad (6)$$

where $\mathcal{D}_{\alpha\gamma} = S_{\gamma\alpha}^\beta \nabla_\beta T$. The trace of the above equation is

$$\mathcal{D} f_{TT} - (R + 2T) f_T + 2f = \kappa^2 \Theta, \quad (7)$$

with $\mathcal{D} = \mathcal{D}^\gamma_\gamma$ and $\Theta = \Theta^\gamma_\gamma$. Equation (6) can be rewritten as

$$G_{\alpha\gamma} = \frac{\kappa^2}{f_T}(\Theta^m_{\alpha\gamma} + \Theta^T_{\alpha\gamma}). \quad (8)$$

The $\Theta^m_{\alpha\gamma}$ constitutes the matter contribution while torsion contribution is presented by

$$\Theta^T_{\alpha\gamma} = \frac{1}{\kappa^2}[-\mathcal{D}_{\alpha\gamma}f_{TT} - \frac{1}{4}g_{\alpha\gamma}(\Theta - \mathcal{D}f_{TT} + Rf_T)]. \quad (9)$$

It can be observed that (6) has an equivalent structure such as $f(R)$ gravity and reduces to GR for $f = T$.

We consider the collapsing star as cylindrical symmetric surface which is characterized by a hypersurface Σ . This timelike $3D$ hypersurface isolates manifold into $4D$ interior and exterior portions. The interior region is taken as cylindrical symmetric collapsing star having line element as follows

$$ds^{2(-)} = X^2(t, r)dt^2 - Y^2(t, r)dr^2 - Z^2(t, r)d\phi^2 - dz^2, \quad (10)$$

where the coordinates t , r , ϕ and z are constrained such as

$$-\infty \leq t \leq \infty, \quad 0 \leq r < \infty, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < z < \infty.$$

in order to preserve cylindrical symmetry. The line element for exterior portion in terms of retarded time coordinate τ and gravitational mass M is given by (Chao-Guang 1995)

$$ds^{2(+)} = \left(-\frac{2M}{r}\right)d\tau^2 + 2drd\tau - r^2(d\phi^2 + \gamma^2 dz^2), \quad (11)$$

where γ is an arbitrary constant and $M = M(\tau)$ represents total mass inside the cylindrical surface. Also, we take the interior region is distributed by locally anisotropic matter for which energy-momentum description is

$$\Theta^m_{\alpha\beta} = (\rho + P_r)U_\alpha U_\beta - P_r g_{\alpha\beta} + (P_z - P_r)V_\alpha V_\beta + (P_\phi - P_r)L_\alpha L_\beta, \quad (12)$$

where $\rho = \rho(t, r)$ denotes the energy density, $P_r = P_r(t, r)$, $P_z = P_z(t, r)$, $P_\phi = P_\phi(t, r)$ represent the corresponding principal pressure components. The four velocity, $U_\alpha = X\delta_\alpha^0$, the four vectors, $L_\alpha = Z\delta_\alpha^2$ and $V_\alpha = \delta_\alpha^3$ satisfy the relations

$$U_\alpha U^\alpha = 1, \quad L_\alpha L^\alpha = 1 = V_\alpha V^\alpha, \quad U^\alpha L_\alpha = L^\alpha V_\alpha = V^\alpha U_\alpha = 0.$$

3 Basic Equations

The choice of tetrad field in $f(T)$ gravity is the key role to setup the framework of $f(T)$ gravity. The bad choice of tetrad are those which constrain the torsion scalar to be constant or vanishes the modification of theory. These tetrad consist in form of diagonal presentation except cartesian symmetry. For spherical and cylindrical symmetry, the good tetrad are non-diagonal tetrad having no restriction for torsion scalar and keeps the modification of the theory. We consider the following tetrad in non-diagonal form for the interior spacetime as

$$h^a{}_\alpha = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & Y \cos \phi & -Z \sin \phi & 0 \\ 0 & Y \sin \phi & Z \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with its inverse

$$h_a{}^\alpha = \begin{pmatrix} \frac{1}{X} & 0 & 0 & 0 \\ 0 & \frac{\cos \phi}{Y} & \frac{\sin \phi}{Y} & 0 \\ 0 & -\frac{\sin \phi}{Z} & \frac{\cos \phi}{Z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The torsion scalar takes the form

$$T = -2 \left[\frac{X'}{XY^2} \left(\frac{Y}{Z} - \frac{Z'}{Z} \right) + \frac{\dot{Y}\dot{Z}}{X^2YZ} \right],$$

where dot and prime denotes derivative with respect to time and radial coordinate. The corresponding field equations are

$$\frac{X^2}{Y^2} \left(\frac{Y'Z'}{YZ} - \frac{Z''}{Z} \right) + \frac{\dot{Y}\dot{Z}}{YZ} = \frac{X^2\kappa^2}{f_T} \left[\rho + \frac{1}{\kappa^2} \left\{ \frac{Tf_T - f}{2} + \frac{1}{2Y^2} \left(\frac{Y}{Z} - \frac{Z'}{Z} \right) f'_T \right\} \right], \quad (13)$$

$$\frac{\dot{Z}'}{Z} - \frac{X'\dot{Z}}{XZ} - \frac{\dot{Y}Z'}{YZ} = \frac{\dot{Z}}{2Z} \frac{f'_T}{f_T}, \quad (14)$$

$$\frac{\dot{Z}'}{Z} - \frac{X'\dot{Z}}{XZ} - \frac{\dot{Y}Z'}{YZ} = \frac{1}{2} \left(\frac{Z'}{Z} - \frac{Y}{Z} \right) \frac{\dot{T}}{T'} \frac{f'_T}{f_T}, \quad (15)$$

$$\frac{Y^2}{X^2} \left(\frac{\dot{X}\dot{Z}}{XZ} - \frac{\ddot{Z}}{Z} \right) + \frac{X'Z'}{XZ} = \frac{Y^2\kappa^2}{f_T} \left[P_r + \frac{1}{\kappa^2} \left\{ \frac{f - Tf_T}{2} - \frac{\dot{Z}\dot{T}}{2X^2ZT'} f'_T \right\} \right], \quad (16)$$

$$\frac{Z^2}{XY} \left[\frac{X''}{Y} - \frac{\ddot{Y}}{X} + \frac{\dot{X}\dot{Y}}{X^2} - \frac{X'Y'}{Y^2} \right] = \frac{Z^2\kappa^2}{f_T} \left[P_\phi + \frac{1}{\kappa^2} \left\{ \frac{f - Tf_T}{2} - \left(\frac{\dot{Y}\dot{T}}{2X^2YT'} - \frac{X'}{2XY^2} \right) f'_T \right\} \right], \quad (17)$$

$$\begin{aligned} \frac{1}{Y^2} \left[\frac{X''}{X} - \frac{X'Y'}{XY} + \frac{X'Z'}{XZ} + \frac{Z''}{Z} - \frac{Y'Z'}{YZ} \right] + \frac{1}{X^2} \left[-\frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} - \frac{\ddot{Z}}{Z} + \frac{\dot{X}\dot{Z}}{XZ} - \frac{\dot{Y}\dot{Z}}{YZ} \right] &= \frac{\kappa^2}{f_T} \left[P_z + \frac{1}{\kappa^2} \left\{ \frac{f - Tf_T}{2} - \left(\frac{\dot{T}}{2X^2T'} \left(\frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) - \frac{1}{2Y^2} \left(\frac{X'}{X} - \frac{Y}{Z} + \frac{Z'}{Z} \right) \right) f'_T \right\} \right]. \end{aligned} \quad (18)$$

We find a relationship using Eqs.(14) and (15) as follows

$$\frac{\dot{Z}}{Z} = \frac{\dot{T}}{T'} \left(\frac{Z'}{Z} - \frac{Y}{Z} \right). \quad (19)$$

In order to match interior and exterior regions of cylindrical symmetric collapsing star, we use junction conditions defined by Darmois. For this purpose, we consider the C-energy, i.e., mass function representing matter inside the cylinder is given by (Throne 1965)

$$m(t, r) = \frac{l}{8} \left(1 - \frac{1}{l^2} \nabla^\alpha \hat{r} \nabla_\alpha \hat{r} \right) = E(t, r), \quad (20)$$

where E is the gravitational energy per unit specific length l of the cylinder. The areal radius \hat{r} is defined as $\hat{r} = \mu l$ where circumference radius has the relation $\mu^2 = \xi_{(1)i} \xi_{(1)}^i$ while $l^2 = \xi_{(2)i} \xi_{(2)}^i$. The terms $\xi_{(1)} = \frac{\partial}{\partial \phi}$ and $\xi_{(2)} = \frac{\partial}{\partial z}$ are the Killing vectors corresponding to cylindrical systems. For Eq.(10), the C-energy turns out

$$m(t, r) = \frac{l}{8} \left[1 + \left(\frac{\dot{Z}}{X} \right)^2 - \left(\frac{Z'}{Y} \right)^2 \right]. \quad (21)$$

The continuity of Darmois conditions (junction conditions) establishes the following constraints

$$\frac{l}{8} \stackrel{\Sigma^{(e)}}{=} m(t, r) - M, \quad l \stackrel{\Sigma^{(e)}}{=} 4C, \quad -p_r \stackrel{\Sigma^{(e)}}{=} \frac{\Theta_{11}^T}{Y^2} - \frac{\Theta_{01}^T}{XY} \quad (22)$$

where $\Sigma^{(e)}$ indicates the outer region for measurements with $r = r_{\Sigma^{(e)}} = \text{constant}$. The conservation of total energy of a system is obtained from Bianchi identities through dynamical equations. The dynamical equations in the framework of $f(T)$ gravity are

$$\left(\Theta^{\alpha\beta} + \Theta^{\alpha\beta T} \right)_{;\beta} U_\alpha = 0, \quad \left(\Theta^{\alpha\beta} + \Theta^{\alpha\beta T} \right)_{;\beta} \chi_\alpha = 0. \quad (23)$$

Using these equations, the dynamical equations for the cylindrically symmetric collapsing star become

$$\dot{\rho} + \frac{\dot{Y}}{Y}(\rho + P_r) + \frac{\dot{Z}}{Z}(\rho + P_\phi) + \frac{J_0}{\kappa^2} = 0, \quad (24)$$

$$P_r' + \frac{X'}{X}(\rho + P_r) + \frac{Z'}{Z}(P_r - P_\phi) + \frac{J_1}{\kappa^2} = 0, \quad (25)$$

where

$$\begin{aligned} J_0 &= X^2 \left\{ \frac{1}{X^2} \left(\frac{Tf_T - f}{2} + \frac{1}{2Y^2} \left(\frac{Y}{Z} - \frac{Z'}{Z} \right) f_T' \right) \right\}_{,0} + X^2 \left\{ \frac{\dot{Z}}{2X^2Y^2Z} f_T' \right\}_{,1} \\ &+ \frac{\dot{X}}{X} (Tf_T - f) + \left\{ \frac{1}{YZ} \left(\frac{\dot{X}}{X} - \frac{\dot{X}Z'}{XY} + \frac{3X'\dot{Z}}{2XY} + \frac{\dot{Y}}{2Y} + \frac{\dot{Z}}{2Z} - \frac{\dot{Y}Z'}{2Y^2} \right. \right. \\ &+ \left. \left. \frac{Y'\dot{Z}}{2Y^2} \right) + \left(\frac{X'Z'}{2XY^2Z} - \frac{X'Y}{2XY^2Z} \right) \frac{\dot{T}}{T'} \right\} f_T', \\ J_1 &= Y^2 \left\{ \frac{1}{2X^2Y^2} \left(\frac{Z'}{Z} - \frac{Y}{Z} \right) \frac{\dot{T}}{T'} f_T' \right\}_{,0} + Y^2 \left\{ \frac{1}{Y^2} \left(\frac{f - Tf_T}{2} - \frac{\dot{Z}\dot{T}}{2X^2ZT'} f_T' \right) \right\}_{,1} \\ &+ \frac{Y'}{Y} (f - Tf_T) + \left\{ \left(\frac{3\dot{X}Z'}{2X^2YZ} - \frac{\dot{Y}}{X^2Z} - \frac{Y'\dot{Z}}{X^2YZ} + \frac{X'}{2XYZ} - \frac{X'Z'}{2XY^2Z} \right. \right. \\ &+ \left. \left. \frac{\dot{X}Z'}{2X^3Z} - \frac{Y\dot{X}}{2X^3Z} - \frac{Y\dot{Z}}{2X^2Z^2} - \frac{X'Z'}{2X^2Z} \right) \frac{\dot{T}}{T'} - \frac{X'Z'}{XY^2Z} + \frac{\dot{Y}\dot{Z}}{2X^2YZ} \right\} f_T'. \end{aligned}$$

In order to develop the collapse equation representing the dynamics of cylindrically symmetric collapsing star in the framework of $f(T)$ gravity, the

choice of $f(T)$ model keeps central importance. Here we assume model in form of power-law upto quadratic torsion scalar term to discuss the instability ranges of cylindrical symmetric collapsing star. This is given by (Sharif and Rani 2014, 2015; Jawad and Rani 2015)

$$f(T) = T + \omega T^2, \quad (26)$$

where ω is an arbitrary constant. This particular model is analogues to a viable model from $f(R)$ gravity such as $f(R) = R + \lambda R^2$ which reduces to GR by taking $\lambda \rightarrow 0$. In $f(R)$ gravity, this model takes part to present dynamics of collapsing star and gives instability ranges for different regimes under many scenarios (Sharif and Kausar 2011, Kausar and Noureen 2014). The power-law $f(T)$ model is very simple form to be considered which provides direct comparison with GR by choosing ω as zero. The finite time singularities are also discussed for power-law model of the type T^m which results the vanishing of these singularities for $m > 1$ (Bamba et al. 2012). Also, this model leads to the accelerated expansion of the universe in phantom phase, possibility of realistic wormhole solutions, solar system tests and instability conditions for a collapsing star. In order to discuss the dynamical instability ranges in the underlying scenario, we impose the condition of static equilibrium (dependent of radial coordinate only) on metric, matter as well as effective parts of the system initially. After some time t , these parts also become time dependent along with r dependency. We represent this strategy by linear perturbation strategy to construct the dynamical equations in order to explore instability ranges for the cylindrically symmetric collapsing star. These perturbations are described as follows

$$X(t, r) = x_0(r) + \varepsilon \Lambda(t) \hat{x}(r), \quad (27)$$

$$Y(t, r) = y_0(r) + \varepsilon \Lambda(t) \hat{y}(r), \quad (28)$$

$$Z(t, r) = z_0(r) + \varepsilon \Lambda(t) \hat{z}(r), \quad (29)$$

$$\rho(t, r) = \rho_0(r) + \varepsilon \hat{\rho}(t, r), \quad (30)$$

$$P_r(t, r) = p_{r0}(r) + \varepsilon \hat{p}_r(t, r), \quad (31)$$

$$P_\phi(t, r) = p_{\phi 0}(r) + \varepsilon \hat{p}_\phi(t, r), \quad (32)$$

$$P_z(t, r) = p_{z0}(r) + \varepsilon \hat{p}_z(t, r), \quad (33)$$

$$m(t, r) = m_0(r) + \varepsilon \hat{m}(t, r), \quad (34)$$

$$T(t, r) = T_0(r) + \varepsilon \Lambda(t) e(r). \quad (35)$$

That is, the quantities with zero subscript refer zero order perturbation of corresponding functions while $0 < \varepsilon \ll 1$. The $f(T)$ model under perturbation of torsion scalar becomes

$$f(T) = T_0(1 + \omega T_0) + \varepsilon \Lambda e(1 + 2\omega T_0), \quad f_T(T) = 1 + 2\omega T_0 + 2\varepsilon \delta \Lambda e. \quad (36)$$

4 Collapse Equation

Here, we construct the collapse equation using $f(T)$ model along with perturbation scheme for the underlying scenario. The static configuration of torsion scalar and mass function with $Z_0 = r$ are given as follows

$$T_0 = -\frac{2x'_0(y_0 - 1)}{rx_0y_0}, \quad m_0 = \frac{l}{8} \left(1 - \frac{1}{y_0^2}\right),$$

while perturbed configuration is given by

$$\begin{aligned} e &= -\frac{2}{rx_0y_0} \left[x'_0 \left(\hat{y} - \hat{z}' + \frac{\hat{z}}{r}(1 - y_0) \right) + (y_0 - 1) \left(\hat{x}' - \frac{x'_0}{x_0y_0}(\hat{x}y_0 + \hat{y}x_0) \right) \right], \\ \hat{m} &= -\frac{\Lambda l}{4y_0^2} \left(\hat{z}' - \frac{\hat{y}}{y_0} \right), \end{aligned}$$

respectively. The dynamical equations take part in order to construct collapse equation for the dynamical instability ranges of cylindrical collapsing star. For this purpose, the non-static configuration of second dynamical equation (25) takes the form

$$\begin{aligned} \hat{p}'_r + \frac{x'_0}{x_0}(\hat{\rho} + \hat{p}_r) + \left(\frac{\hat{x}'}{x_0} - \frac{\hat{x}x'_0}{x_0} \right)(\rho_0 + p_{r0})\Lambda + \frac{1}{r}(\hat{p}_r - \hat{p}_\phi) \\ + \frac{\hat{z}'}{r}(p_{r0} - p_{\phi0})\Lambda + \frac{J_{1p}}{\kappa^2} = 0, \end{aligned} \quad (37)$$

where

$$\begin{aligned} J_{1p} &= \frac{e\omega T_0^{2'}}{rx_0^2}(1 - y_0)\ddot{\Lambda} + 2y_0\hat{y}\left(\frac{\omega T_0^2}{2y_0^2}\right)_{,1}\Lambda + y_0^2\left\{\frac{1}{y_0^2}\left(\frac{\hat{y}\omega T_0^2}{y_0}\right.\right. \\ &\quad \left.\left.- \omega e T_0\right)\right\}_{,1}\Lambda - \frac{2\omega e T_0 y'_0}{y_0}\Lambda - \frac{1}{y_0}\left(\hat{y}' - \frac{y'_0\hat{y}}{y_0}\right)\omega T_0^2\Lambda + \frac{e\omega T_0^{2'}}{rx_0}\left(\frac{x'_0}{y_0}\right. \end{aligned}$$

$$- \frac{\hat{x}'}{y_0^2} - \frac{\hat{x}'}{x_0} \Big) - \frac{2\hat{x}'\omega e'}{rx_0y_0^2} \Lambda - 2\omega T_0' \left(\frac{\hat{x}}{x_0} + \frac{\hat{y}}{y_0} + \frac{\hat{z}}{r} \right) \Lambda.$$

Equation (37) is the general collapse equation which depicts the instability of hydrostatic equilibrium of cylindrical gravitating fluid in $f(T)$ gravity. To analyze the instability of fluid using this equation, we need the expressions for $\hat{\rho}$, \hat{p}_r , \hat{p}_ϕ and Λ through basic equations of the underlying system.

Applying perturbation strategy on first dynamical equation, the perturbed part is given as follows

$$\dot{\hat{\rho}} + \left[\frac{\hat{y}}{y_0}(\rho_0 + p_{r0}) + \frac{\hat{z}}{r}(\rho_0 + p_{\phi 0}) + \frac{J_{0p}}{\kappa^2} \right] \dot{\Lambda} = 0, \quad (38)$$

where

$$\begin{aligned} J_{0p} &= e\omega T_0 + \frac{\omega e'(y_0 - 1)}{ry_0^2} + \frac{\omega T_0'}{ry_0^2} \left(\hat{y} - \hat{z}' + \frac{\hat{z}}{r}(1 - y_0) - \frac{2\hat{y}(y_0 - 1)}{y_0} \right) \\ &- \frac{\hat{x}}{x_0} \left(\omega T_0^2 + \frac{2\omega T_0'(y_0 - 1)}{ry_0^2} \right) + x_0^2 \left(\frac{z\omega T_0'}{rx_0^2y_0^2} \right) + \frac{\hat{x}\omega T_0^2}{x_0} + \frac{2\omega T_0'}{ry_0} \\ &\times \left(\frac{\hat{x}}{x_0} - \frac{\hat{x}}{x_0y_0} + \frac{\hat{z}x_0'}{2x_0y_0} + \frac{\hat{y}}{2y_0} + \frac{\hat{z}}{2r} - \frac{\hat{y}}{2y_0^2} + \frac{\hat{z}x_0'}{x_0y_0} + \frac{\hat{z}y_0'}{2y_0^2} \right. \\ &\left. + \frac{ex_0'T_0'(1 - y_0)}{2x_0y_0} \right) \end{aligned}$$

Integrating Eq.(38) with respect to time, we obtain the non-static energy density which is given by

$$\hat{\rho} = - \left[\frac{\hat{y}}{y_0}(\rho_0 + p_{r0}) + \frac{\hat{z}}{r}(\rho_0 + p_{\phi 0}) + \frac{J_{0p}}{\kappa^2} \right] \Lambda. \quad (39)$$

The Harrison-Wheeler equation of state establishes a relationship between energy density and pressure such as (Harrison et al. 1965)

$$\hat{p}_i = \Gamma \frac{p_{i0}}{\rho_0 + p_{i0}} \hat{\rho}, \quad (40)$$

where Γ is called adiabatic index. We use this index in order to examine instability ranges in the context of $f(T)$ gravity. The adiabatic index Γ finds the rigidity of the fluid and evaluates the change of pressure to corresponding

density. Substituting the value of $\hat{\rho}$ from Eq.(39) in (40) for \hat{p}_r , and \hat{p}_ϕ , it yields

$$\hat{p}_r = -\Lambda \left[\frac{\hat{y}}{y_0} p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p} \right] \Gamma, \quad (41)$$

$$\hat{p}_\phi = -\Lambda \left[\frac{\hat{y}}{y_0} \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\phi 0}} p_{\phi 0} + \frac{c}{r} p_{\phi 0} + \frac{1}{\kappa^2} \frac{p_{\phi 0}}{\rho_0 + p_{\phi 0}} J_{0p} \right] \Gamma. \quad (42)$$

To find out the value of $\Lambda(t)$, we perturbed field equation (16) and its non-static part is given by

$$\begin{aligned} -\frac{\hat{z}}{r x_0^2} \ddot{\Lambda} + \frac{1}{r x_0 y_0^2} \left(-\frac{2\hat{y}x'_0}{y_0} - \frac{\hat{x}x'_0}{x_0} - \frac{\hat{z}x'_0}{r} + \hat{x}' + \hat{z}'x'_0 \right) \Lambda = \\ -\frac{2\omega e \kappa^2 \Lambda}{(1 + 2\omega T_0)^2} p_{r0} + \frac{\kappa^2 \hat{p}_r}{1 + 2\omega T_0} - \frac{e\omega T_0 \Lambda}{1 + 2\omega T_0} - \frac{e\omega^2 T_0^2 \Lambda}{(1 + 2\omega T_0)^2}. \end{aligned} \quad (43)$$

We use the last junction condition given in Eq.(22) with $r = r_{\Sigma^{(e)}} = \text{constant}$ which under perturbation strategy yields

$$p_{r0} \stackrel{\Sigma^{(e)}}{=} \frac{\omega T_0^2}{2\kappa^2}, \quad \hat{p}_r \stackrel{\Sigma^{(e)}}{=} \frac{\omega e T_0}{\kappa^2} \Lambda. \quad (44)$$

Using this equation along with $r = r_{\Sigma^{(e)}} = \text{constant}$ in Eq.(43), we obtain

$$\ddot{\Lambda} - \sigma_{\Sigma^{(e)}} \Lambda \stackrel{\Sigma^{(e)}}{=} 0, \quad \text{where} \quad \sigma_{\Sigma^{(e)}} = \frac{2r_{\Sigma^{(e)}} \omega^2 e T_0^2 x_0^2}{\hat{z}(1 + 2\omega T_0)^2}.$$

Its solution is

$$\Lambda(t) = c_1 e^{\sqrt{\sigma_{\Sigma^{(e)}}} t} + c_2 e^{-\sqrt{\sigma_{\Sigma^{(e)}}} t},$$

representing stability and instability phases of cylindrical gravitating fluid through static and non-static parts and c_1, c_2 are constants. We assume the hydrostatic equilibrium for which Λ is zero at large past time, $t = -\infty$. After this scenario with the evolution of time, the system recruits into present phase, reduce its areal radius and commences to collapse. In regards to discuss instability analysis of cylindrical collapsing star, we take only static solution ($\Lambda(-\infty) = 0$) which gives $c_2 = 0$. Choosing $c_1 = -1$, we get

$$\Lambda(t) = -e^{\sqrt{\sigma_{\Sigma^{(e)}}} t}, \quad \text{where} \quad \sigma_{\Sigma^{(e)}} > 0. \quad (45)$$

Inserting all the corresponding values in general collapse equation (37), we obtain

$$\begin{aligned}
& \Gamma \left[\left\{ \frac{\hat{y}}{y_0} p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p} \right\}_{,1} + \frac{x'_0}{x_0} \left\{ \frac{\hat{y}}{y_0} p_{r0} \right. \right. \\
& + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p} \left. \right\} + \frac{1}{r} \left\{ \frac{\hat{y}}{y_0} p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} \right. \\
& + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p} - \left(\frac{\hat{y}}{y_0} \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\phi 0}} p_{\phi 0} + \frac{c}{r} p_{\phi 0} + \frac{1}{\kappa^2} \frac{p_{\phi 0}}{\rho_0 + p_{\phi 0}} J_{0p} \right) \left. \right\} \left. \right] \\
& + \frac{x'_0}{x_0} \left\{ \frac{\hat{y}}{y_0} (\rho_0 + p_{r0}) + \frac{\hat{z}}{r} (\rho_0 + p_{\phi 0}) + \frac{J_{0p}}{\lambda^2} \right\} - \left(\frac{\hat{x}'}{x_0} - \frac{\hat{x} x'_0}{x_0} \right) (\rho_0 + p_{r0}) \\
& - \frac{\hat{z}'}{r} (p_{r0} - p_{\phi 0}) + \frac{J_{1p}}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}} t}} = 0. \tag{46}
\end{aligned}$$

This equation represents the collapse equation for cylindrically gravitating object in the framework of $f(T)$ gravity.

5 Dynamical Instability Analysis

In this section, we analyze the instability ranges of cylindrically symmetric self-gravitating fluid in $f(T)$ gravity using Newtonian and post-Newtonian conditions in the collapse equation for corresponding regimes with the help of adiabatic index.

5.1 Newtonian Regime

For Newtonian regime, we have the constraints as follows

$$x_0 = 1 = y_0, \quad \rho_0 \geq p_{r0}, \quad \rho_0 \geq p_{\phi 0}.$$

These constraints implies that $x'_0 = 0 = y'_0$, $\frac{p_{r0}}{\rho_0 + p_{r0}} \rightarrow 0$ and $\frac{p_{\phi 0}}{\rho_0 + p_{\phi 0}} \rightarrow 0$. Using these expressions in Eq.(46) and consequently collapse equation reduces to

$$\begin{aligned}
& \Gamma \left[\left(\hat{y} p_{r0} + \frac{\hat{z}}{r} p_{r0} \right)_{,1} + \frac{1}{r} \left(\hat{y} + \frac{\hat{z}}{r} \right) (p_{r0} - p_{\phi 0}) \right] - \hat{x}' \rho_0 - \frac{\hat{z}'}{r} (p_{r0} - p_{\phi 0}) \\
& + \frac{J_{1p}^N}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}} t}} = 0 \tag{47}
\end{aligned}$$

which represents the cylindrically symmetric self-gravitating fluid in hydrostatic equilibrium. Here J_{1p}^N expresses Newtonian approximation terms in J_{1p} , i.e., those terms remained after applying above constraints. The system turns to collapses or unstable if

$$\Gamma < \frac{\hat{x}'\rho_0 + \frac{\hat{z}'}{r}(p_{r0} - p_{\phi 0}) - \frac{J_{1p}^N}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}}t}}}{\left(\hat{y}p_{r0} + \frac{\hat{z}}{r}p_{r0}\right)_{,1} + \frac{1}{r}\left(\hat{y} + \frac{\hat{z}}{r}\right)(p_{r0} - p_{\phi 0})} = \frac{A_N}{B_N}. \quad (48)$$

It is noted that we have assigned numerator and denominator as A_n and B_n for the sake for simplicity. We assume adiabatic index to be positive throughout the scenario to maintain difference between gradient of principal pressure components and gravitational forces. Under this condition, the left hand side of above inequality depending on dynamical properties such as density, pressure components and torsion terms remains positive. The system remains unstable as long as this inequality holds. Thus it leads to following possibilities

- I: $\hat{x}'\rho_0 + \frac{\hat{z}'}{r}(p_{r0} - p_{\phi 0}) - \frac{J_{1p}^N}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}}t}} = \left(\hat{y}p_{r0} + \frac{\hat{z}}{r}p_{r0}\right)_{,1} + \frac{1}{r}\left(\hat{y} + \frac{\hat{z}}{r}\right)(p_{r0} - p_{\phi 0})$ or equivalently $A_N = B_N$
- II: $\hat{x}'\rho_0 + \frac{\hat{z}'}{r}(p_{r0} - p_{\phi 0}) - \frac{J_{1p}^N}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}}t}} < \left(\hat{y}p_{r0} + \frac{\hat{z}}{r}p_{r0}\right)_{,1} + \frac{1}{r}\left(\hat{y} + \frac{\hat{z}}{r}\right)(p_{r0} - p_{\phi 0})$ or equivalently $A_N < B_N$
- III: $\hat{x}'\rho_0 + \frac{\hat{z}'}{r}(p_{r0} - p_{\phi 0}) - \frac{J_{1p}^N}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}}t}} > \left(\hat{y}p_{r0} + \frac{\hat{z}}{r}p_{r0}\right)_{,1} + \frac{1}{r}\left(\hat{y} + \frac{\hat{z}}{r}\right)(p_{r0} - p_{\phi 0})$ or equivalently $A_N > B_N$

In the case I, we obtain $\Gamma < \frac{A_N}{B_N} = 1$ which implies that $0 < \Gamma < 1$, the instability range for cylindrical system in Newtonian range. For the case II, the condition $A_N < B_N$ yields $\Gamma < \frac{A_N}{B_N} < 1$ which leads to the instability range as $0 < \Gamma < \frac{A_N}{B_N}$ where $\frac{A_N}{B_N} < 1$. If $A_N > B_N$ holds, we obtain the instability range by the inequality as $0 < \Gamma < \frac{A_N}{B_N}$ where $\frac{A_N}{B_N} > 1$. It is noted that, we may recover the GR in Newtonian limit for instability range as $\Gamma < \frac{4}{3}$ in this case. This limit also includes the first two instability ranges.

Isotropic Pressure

Here we discuss the instability ranges for the case when all pressure components become equal, i.e., isotropic pressure fluid ($p_r = p_\phi = p_z = p$). For isotropic cylindrical fluid system, Eq.(48) reduces to

$$\Gamma < \frac{\hat{x}'\rho_0 - \frac{J_{1p}^N}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma}(e)}t}}}{(\hat{y}p + \frac{\hat{z}}{r}p)_{,1}}. \quad (49)$$

This inequality indicates the unstable behavior if it holds and expresses the correspondingly instability ranges.

Asymptotic Behavior

Substituting $\omega = 0$ in Eq.(48) implies $J_{1p}^N = 0$, it yields the inequality for anisotropic pressure fluid as

$$\Gamma < \frac{\hat{x}'\rho_0 + \frac{\hat{z}'}{r}(p_{r0} - p_{\phi0})}{\left(\hat{y}p_{r0} + \frac{\hat{z}}{r}p_{r0}\right)_{,1} + \frac{1}{r}\left(\hat{y} + \frac{\hat{z}}{r}\right)(p_{r0} - p_{\phi0})}. \quad (50)$$

The adiabatic index indicates the results for GR as long as above inequality satisfied. For cylindrically symmetric isotropic fluid, we obtain

$$\Gamma < \frac{\hat{x}'\rho_0}{(\hat{y}p + \frac{\hat{z}}{r}p)_{,1}}. \quad (51)$$

In these cases, the instability ranges are retrieved accordingly as obtained for Eq.(48).

5.2 Post-Newtonian Regime

Here we study the dynamical instability of cylindrical self-gravitating object with post-Newtonian limits. These limits are $x_0 = 1 - \frac{m_0}{r}$, $y_0 = 1 + \frac{m_0}{r}$ upto order $O(\frac{m_0}{r})$. Consequently the fluid for hydrostatic equilibrium by inserting x_0 and y_0 in collapse equation takes the form

$$\Gamma \left[\left\{ \hat{y} \left(1 - \frac{m_0}{r} \right) p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p}^{pN} \right\}_{,1} + \frac{m_0}{r^2} \left\{ \hat{y} \left(1 - \frac{m_0}{r} \right) p_{r0} \right.$$

$$\begin{aligned}
& + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p}^{pN} \Big\} + \frac{1}{r} \Big\{ \hat{y} \left(1 - \frac{m_0}{r} \right) p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} \\
& + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p}^{pN} - \left(\hat{y} \left(1 - \frac{m_0}{r} \right) \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\phi 0}} p_{\phi 0} + \frac{c}{r} p_{\phi 0} + \frac{1}{\kappa^2} \frac{p_{\phi 0}}{\rho_0 + p_{\phi 0}} J_{0p}^{pN} \right) \Big\} \Big] \\
& + \frac{m_0}{r^2} \Big\{ \hat{y} (\rho_0 + p_{r0}) + \frac{\hat{z}}{r} (\rho_0 + p_{\phi 0}) + \frac{J_{0p}^{pN}}{\lambda^2} \Big\} - \left(\hat{x}' \left(1 + \frac{m_0}{r} \right) - \frac{\hat{x} m_0}{r^2} \right) (\rho_0 + p_{r0}) \\
& - \frac{\hat{z}'}{r} (p_{r0} - p_{\phi 0}) + \frac{J_{1p}^{pN}}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}} t}} = 0. \tag{52}
\end{aligned}$$

The terms with superscript pN point out the terms with post-Newtonian approximations in corresponding expressions. For dynamical instability, we find

$$\Gamma < \frac{A_{pN}}{B_{pN}}, \tag{53}$$

where

$$\begin{aligned}
A_{pN} &= \left(\hat{x}' \left(1 + \frac{m_0}{r} \right) - \frac{\hat{x} m_0}{r^2} \right) (\rho_0 + p_{r0}) - \frac{m_0}{r^2} \Big\{ \hat{y} (\rho_0 + p_{r0}) - \frac{J_{0p}^{pN}}{\kappa^2} \\
& + \frac{\hat{z}}{r} (\rho_0 + p_{\phi 0}) \Big\} + \frac{\hat{z}'}{r} (p_{r0} - p_{\phi 0}) - \frac{J_{1p}^{pN}}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}} t}}, \\
B_{pN} &= \left\{ \hat{y} \left(1 - \frac{m_0}{r} \right) p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p}^{pN} \right\}_{,1} + \frac{m_0}{r^2} \\
& \times \left\{ \hat{y} p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p}^{pN} \right\} + \frac{1}{r} \left\{ \hat{y} \left(1 - \frac{m_0}{r} \right) \right. \\
& \times p_{r0} + \frac{c}{r} \frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}} p_{r0} + \frac{1}{\kappa^2} \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0p}^{pN} - \left(\hat{y} \left(1 - \frac{m_0}{r} \right) \right. \\
& \times \left. \left. \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\phi 0}} p_{\phi 0} + \frac{c}{r} p_{\phi 0} + \frac{1}{\kappa^2} \frac{p_{\phi 0}}{\rho_0 + p_{\phi 0}} J_{0p}^{pN} \right) \right\}.
\end{aligned}$$

Similar to the case of Newtonian regime, we can develop three possibilities on A_{pN} and B_{pN} , i.e., $A_{pN} = B_{pN}$, $A_{pN} < B_{pN}$, $A_{pN} > B_{pN}$. These possibilities yield the instability ranges as $0 < \Gamma < \frac{A_{pN}}{B_{pN}}$ which contains $0 < \Gamma < 1$.

Isotropic Pressure

Taking isotropy of pressure in Eq.(52), we get the unstable behavior as

$$\Gamma < \frac{A_{ipN}}{B_{ipN}}, \quad (54)$$

where

$$\begin{aligned} A_{ipN} &= \left(\hat{x}' \left(1 + \frac{m_0}{r} \right) - \frac{\hat{x} m_0}{r^2} \right) (\rho_0 + p) - \frac{m_0}{r^2} \left\{ \hat{y} (\rho_0 + p) - \frac{J_{0p}^{pN}}{\kappa^2} \right. \\ &\quad \left. + \frac{\hat{z}}{r} (\rho_0 + p) \right\} - \frac{J_{1p}^{pN}}{\kappa^2} \frac{1}{e^{\sqrt{\sigma_{\Sigma(e)}} t}}, \\ B_{ipN} &= \left\{ \hat{y} \left(1 - \frac{m_0}{r} \right) p_{r0} + \frac{c}{r} p + \frac{1}{\kappa^2} \frac{p}{\rho_0 + p} J_{0p}^{pN} \right\}_{,1} + \frac{m_0}{r^2} \\ &\quad \times \left\{ \hat{y} p + \frac{c}{r} p + \frac{1}{\kappa^2} \frac{p}{\rho_0 + p} J_{0p}^{pN} \right\} + \frac{1}{r} \left\{ \hat{y} \left(1 - \frac{m_0}{r} \right) p + \frac{c}{r} p + \frac{1}{\kappa^2} \right. \\ &\quad \times \left. \frac{p}{\rho_0 + p} J_{0p}^{pN} - \left(\hat{y} \left(1 - \frac{m_0}{r} \right) p + \frac{c}{r} p + \frac{1}{\kappa^2} \frac{p}{\rho_0 + p} J_{0p}^{pN} \right) \right\}. \end{aligned}$$

The stability ranges for the cylindrically symmetric isotropic fluid have the criteria as for inequality (53).

Asymptotic Behavior

In order to recover the results in GR, we insert $\omega = 0$ which leads to $J_{0p}^{pN} = 0 = J_{1p}^{pN}$ for isotropic as well as anisotropic fluids.

6 Conclusion

The dynamical instability of a self-gravitating general relativistic object undergoing gravitational collapse process has become widely considered phenomena in GR as well as in modified theories of gravity. This process stays significantly at the center of structure formation and holds the evolutionary development of these objects. We have taken cylindrically symmetric line element as interior spacetime while exterior spacetime in retarded time coordinate in the framework of $f(T)$ gravity. The locally anisotropic matter distribution is considered for which we have obtained an important result at boundary of matching both interior and exterior regions. We have developed

$f(T)$ field equations along with some basic equations such that dynamical equations through Bianchi identities. In order to get insights in more realistic way, we have assumed a specific power-law $f(T)$ model with linear and quadratic torsion scalar terms. This model is used to discuss many cosmological scenarios as well as general symmetric solutions such as stability of spherical collapsing star and wormhole solutions.

Keeping system in hydrostatic equilibrium initially and then perturbed with the evolution of time by linear perturbation strategy. This strategy is applied on all matter, metric, mass and torsion components. In order to construct collapse equation, the dynamical equations are used in an appropriate way along with adiabatic index Γ . With the help of second law of thermodynamics, this index gives the ratio of specific heat using energy density and pressure components. We have obtained the solution for non-static perturbed quantity using matching conditions which satisfied the initial state of equilibrium. We have applied the constraints of Newtonian and post-Newtonian regimes in collapse equation to find the instability ranges for the cylindrically symmetric collapsing object in the framework of $f(T)$ gravity.

We have found instability ranges for both regimes represented by adiabatic index. We have also found these ranges taking into account isotropic pressure as well as asymptotic behavior, i.e., GR solutions. We have found the instability range as $0 < \Gamma < \frac{A}{B}$ for $\frac{A}{B} = 1$, $\frac{A}{B} < 1$, $\frac{A}{B} > 1$ where A and B represent the corresponding expressions in each case. These expressions depend on matter, metric and torsion terms. This range also admits the GR condition of unstable behavior through adiabatic index which is $\Gamma < \frac{4}{3}$. It is noted that for other forms of $f(T)$ models instead of power-law form, the quantitative consequences are changed while qualitative consequences remain same. For instance, we consider the exponential model, $f(T) = T - \alpha_1 T(1 - e^{\frac{rT_c}{T}})$ for which we may obtain static and perturbed parts using perturbation scheme adopting some more steps. The collapse equation as well as instability ranges through adiabatic index will depend on exponential terms throughout. However, the qualitative consequences remain unchanged due to the fact that all the instability ranges via adiabatic index depend on matter, metric and torsion dependent terms.

The dynamical instability in $f(R)$ gravity has been discussed taking CDTT model for a cylindrically symmetric collapsing star (Kausar 2013). It has been found that adiabatic index depends on immense perturbed terms of this model along with some positivity constraints for the dynamical unsta-

ble behavior. In Brans-Dicke theory of gravity (Sharif & Manzoor 2015), the instability ranges of a collapsing stellar object having cylindrically symmetry have been investigated. It yields the instability ranges through collapse equation depend on the dynamical variables of collapsing fluid. The ranges for unstable behavior are obtained as $0 < \Gamma < 1$ while for a special case, it gives $\Gamma > 1$. In $f(T)$ gravity, we have found dynamical instability ranges for spherically symmetric collapsing star with and without expansion as well as with shear-free conditions taking anisotropic fluid (Sharif & Rani 2014, 2015; Jawad & Rani 2015). However, in the present paper, we have analyzed dynamical unstable behavior taking cylindrical symmetric object which gives less complexity in expressions A and B . Also, we have reduced the results in the limit of isotropic fluid distribution and to GR limit.

Chandrasekhar (1964) was the first who explored dynamical instability ranges of a spherically symmetric isotropic fluid in GR. He established these ranges through adiabatic index which depends on its numerical value. That is, the weight of the outer layer increases rapidly as compared to the pressure in a star for $\Gamma < \frac{4}{3}$ which leads to the unstable behavior of the star. Whereas, for $\Gamma > \frac{4}{3}$, the pressure overcomes the weight of out layers and yields the stability of the star. In $f(T)$ gravity, Sharif & Rani (2014, 2015) analyzed the dynamics of self-gravitating object with spherical symmetry via expansion and expansion free cases. Jawad & Rani (2015) examined the instability ranges taking into account shear-free condition for Newtonian and post-Newtonian regimes via adiabatic index. These works are only discussed for anisotropic fluid. In order to compare the results of present paper, we analyze that results depend on the physical quantities, like energy density, pressure, curvature terms and mass of the cylinder. However, to make a correspondence with the results of isotropic sphere, we have established possibilities (after Eq.(48) as I, II, III) on these physical quantities in each case (isotropic as well as anisotropic fluids) to have numerical results like Chandrasekhar (1964).

So far we know that the cosmographic features of $f(T)$ gravity mimics LCDM model as well as phantom dark energy models. Although there are some ambiguities about the validity of solar system tests for $f(T)$ due to the absence of the Schwarzschild solution as the vacuum (Rodrigues et al. 2013), but the dynamics of stellar objects and what we studied here as the (in)stability of cylindrical objects provides a good sample to check the validity of $f(T)$ gravity in the astrophysics. Briefly the stability conditions were obtained in our paper have significant physical meaning in comparison to

the classical results. Furthermore, the model which we studied here, the torsion based version of the Starobinsky model tested among several types of models with cosmological data, so we believe that our paper will be useful for astrophysical tests of compact objects in $f(T)$ gravity.

References

- Bamba, K., Geng, C-Q., Lee, C-C., 2010, arXiv:1008.4036
 Bamba, K., Geng, C-Q., Lee, C-C., Luo, L-W., 2011, JCAP 1101, 021
 Bamba, K., et al., 2012, Phys. Rev. D 85, 104036
 Bamba, K., Nojiri, S., Odintsov, S.D., 2014, Phys. Lett. B 731, 257
 Chan, R., Herrera, L., Santos, N.O., 1993, Mon. Not. R. Astron. Soc. 265, 533
 Chandrasekhar, S., 1964, Astrophys. J. 140, 417
 Chao-Guang, H., 1995, Acta Phys. Sin. 4, 617
 de la Cruz-Dombriz, A., Dunsby, P.K.S., Saez-Gomez, D., 2014, JCAP 1412, 048
 Daouda, M.H., Rodrigues, M.E., Houndjo, M.J.S., 2012, Eur. Phys. J. C 72, 1890
 Daouda, M.H., Rodrigues, M.E. and Houndjo, M.J.S., 2012, Eur. Phys. J. C. 71, 1817
 Ferraro, R., Fiorini, F.: Phys. Rev. D **75**(2007)084031; *ibid.* **78**(2008)124019.
 Gonzalez, P.A., Saridakis, E.N., Vsquez, Y., 2012, JHEP 53, 2012
 Harrison, B.K. et al., 1965, *Gravitation Theory and Gravitational Collapse* (Univ. of Chicago Press, 1965)
 Herrera, L., Santos, N.O., 1995, Astrophys. J. 438, 308
 Herrera, L., Le Denmat, G., Santos, N.O., 2012, Gen. Relativ. Gravit. 44, 1143
 Herrera, L., Santos, N.O., Le Denmat, G., 1989, Mon. Not. Roy. Astron. Soc. 237, 257
 Houndjo, M.J.S., Momeni, D., Myrzakulov, R., 2012, Int. J. Mod. Phys. D 21, 1250093
 Jamil, M., Momeni, D., Myrzakulov, R., 2013, Eur. Phys. J. C 73, 2267
 Jamil, M., Momeni, D., Myrzakulov, R., 2012, Eur. Phys. J. C 72, 1959
 Jamil, M., Momeni, D., Raza, M., Myrzakulov, R., 2012, Eur. Phys. J. C 72, 1999

Jawad, A., Rani, S., 2015, Eur. Phys. J. C 75, 173
 Jawad, A., Rani, S., 2015, Eur. Phys. J. C 75, 548
 Kausar, H.R., Noureen, I., 2014, Eur. Phys. J. C 74, 2760
 Kausar, H.R., Noureen, I. and Shahzad, M.U.: Eur. Phys. J. Plus **130**(2015)204.
 Kausar, H.R., 2013, JCAP 01, 007
 Li, B., Sotiriou, T.P., John D. Barrow, J.D., 2011, Phys. Rev. D 83, 064035
 Li, M., Miao, R-X., Miao, Y-G., 2011, JHEP 011, 108
 Linder E.V., 2010a, Phys. Rev. D 81, 127301, Erratum: (2010b) ibid. 82, 109902
 Nashed, G.G.L, 2015, Adv. High Energy Phys. 2015, 680457
 Roshan, M., Abbass, S., 2015, Astrophys. J. 802, 9
 Roshan, M., Abbass, S., 2014, Phys. Rev. D 90, 044010
 Sharif, M, Kausar, H.R., 2011, JCAP 07, 022
 Sharif, M, Manzoor, R., 2016, accepted for publication in J. Exp. Theor. Phys.
 Sharif, M, Manzoor, R., 2014, Astrophys. Space Sci. 354, 2122
 Sharif, M, Rani, S., 2014a, Eur. Phys. J. Plus 129, 237
 Sharif, M, Rani, S., 2014b, Adv. High Energy Phys. 2014, 691497
 Sharif, M, Rani, S., 2015, Int. J. Theor. Phys. 54, 2524
 Sharif, M, Rani, S., 2014, Mon. Not. Roy. Astron. Soc. 440, 2255
 Sharif, M, Rani, S., 2013, Phys. Rev. D 88, 123501
 Skripkin, V.A., 1960, Soviet Phys. Doklady 135, 1183
 Throne, K.S., 1965, Phys. Rev. B 138, 251
 Wang, T., 2011a, Phys. Rev. D 84, 024042
 Wang, T., 2011b, JCAP 11, 033
 Wu, P., Yu, H., 2011, Eur. Phys. J. C 71, 1552
 Wu, P., Yu, H., 2010a, Phys. Lett. B 692, 176
 Wu, P., Yu, H., 2010b, Phys. Lett. B 693, 415